Reachability Problems for Continuous Linear Dynamical Systems

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(Joint work with Ventsislav Chonev and Joël Ouaknine)

CONCUR
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Reachability for Continuous-Time Markov Chains

Distribution $P(t)$ at time $t$ satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

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is the **rate matrix**.

“Is it ever more likely to be a Bear market than a Bull market?”

$$\exists t \ (P(t)_{Bear} \geq P(t)_{Bull})$$
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is the rate matrix.

Stationary distribution $\pi = (0.885, 0.071, 0.044)$. 
“To analyze a cyber-physical system, such as a pacemaker, we need to consider the discrete software controller interacting with the physical world, which is typically modelled by differential equations”

Rajeev Alur (CACM, 2013)
Hybrid automaton $= \text{states} + \text{variables } x \in \mathbb{R}^k$
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- $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow \quad \text{timed automata}$
Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$

- $\dot{\mathbf{x}} = 1 \Rightarrow$ timed automata
- $\dot{\mathbf{x}} = \mathbf{c} \Rightarrow$ rectangular hybrid automata
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Is this location a trap?

$\dot{\mathbf{x}} = 3\mathbf{x} - \mathbf{y}$

$\dot{\mathbf{y}} = \mathbf{x} - 5\mathbf{y}$

$\mathbf{x} := 2$

$\mathbf{y} := 4$

$\mathbf{x} \geq 10 \land \mathbf{y} \leq 2?$
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- o-minimal flows + strong resets $\Rightarrow$ reachability decidable
Hybrid Automata: Various Continuous Dynamics

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Is this location a trap?

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\begin{aligned}
x &:= 2 \\
y &:= 4 \\
\dot{x} &= 3x - y \\
\dot{y} &= x - 5y \\
x &\geq 10 \\
y &\leq 2?
\end{aligned}
\]
Reachability for Continuous Linear Dynamical Systems

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Is ever more likely to be a Bear market than a Bull market:

\( \exists t \left( P(t)_{\text{Bear}} \geq P(t)_{\text{Bull}} \right) ? \)
\( x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^k \)
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Reachability for Continuous Linear Dynamical Systems

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\[ f^{(k)}(t) + a_{k-1}f^{(k-1)}(t) + \ldots + a_1 f'(t) + a_0 f(t) = 0 \]
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\[
f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}
\]
Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be given as above, with all coefficients algebraic.
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**BOUNDDED-ZERO Problem**

**Instance:** $f$ and bounded interval $[a, b]$

**Question:** Is there $t \in [a, b]$ such that $f(t) = 0$?

• Decidability open! [Bell, Delvenne, Jungers, Blondel 2010]
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A lot of work since 1920s on the zeros of exponential polynomials

\[ f(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z} \]

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CONTINUOUS-ORBIT Problem

The problem of whether the trajectory \( x(t) = e^{At} x(0) \) reaches a given target point was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).
Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.


In dimension 3, BOUNDED-ZERO and ZERO are decidable.


Assuming Schanuel’s Conjecture, BOUNDED-ZERO is decidable in all dimensions.

It turns out that this result (in fact, a powerful generalisation of it) had already been discovered (but never published) in the early 1990s by Macintyre and Wilkie!

[Angus Macintyre, personal communication, July 2015]
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Reachability for Continuous Linear Dynamical Systems

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In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.


In dimension 9 (and above), decidability of ZERO would entail major breakthroughs in Diophantine approximation—the Diophantine approximation type of $\alpha$ would be computable to within arbitrary precision.
Schanuel’s Conjecture

Theorem (Lindemann-Weierstrass)

If \( a_1, \ldots, a_n \) are algebraic numbers linearly independent over \( \mathbb{Q} \), then \( e^{a_1}, \ldots, e^{a_n} \) are algebraically independent.
Schanuel’s Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

Schanuel’s Conjecture

If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$ then some $n$-element subset of $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$ is algebraically independent.
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Example

By Schanuel’s conjecture some two-element subset of \( \{1, \pi i, e^1, e^{\pi i} \} \) is algebraically independent.
Real-valued exponential polynomial \( f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t} \)
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Real-valued exponential polynomial \( f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t} \)

\[ f(t) \]

'non-trivial' zero \( \Rightarrow t^* \) transcendental
The BOUNDED-ZERO Problem

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Can this situation arise?
Real-valued exponential polynomial \( f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t} \)

Easily! For example, \( f(t) = 2 + e^{it} + e^{-it} \).
Laurent Polynomials and Factorisation

Example

Write \( f(t) = 2 + e^{it} + e^{-it} \) in the form \( f(t) = P(e^{it}) \) for the Laurent polynomial

\[ P(z) = 2 + z + z^{-1}. \]
**Example**

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

  $$P(z) = 2 + z + z^{-1}.$$  

- Factorisation $P(z) = (1 + z)(1 + z^{-1})$ induces a factorisation

  $$f(t) = \underbrace{(1 + e^{it})(1 - e^{it})}_{f_1(t)} \underbrace{(1 - e^{it})}_{f_2(t)}$$
## Laurent Polynomials and Factorisation

### Example

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\[
f(t) = (1 + e^{it}) (1 - e^{it}) = f_1(t) f_2(t)
\]

- **Common zeros of** \( f_1 \) and \( f_2 \) are tangential zeros of \( f \)
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Idea: factorise \( f \).
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- Common zeros of \( f_1 \) and \( f_2 \) are tangential zeros of \( f \)

Idea: factorise \( f \). Noting that factors may be complex-valued!
Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$. 

**Lemma**
Assuming Schanuel's conjecture, if $f$ is real valued and $P$ is irreducible then $f$ has no tangential zeros.

Complex case requires some new ideas . . .
The Real Case

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**Lemma**

Assuming Schanuel’s conjecture, if \( f \) is real valued and \( P \) is irreducible then \( f \) has no tangential zeros.

Complex case requires some new ideas . . .
“there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don’t know we don’t know.”
Continued Fractions

Finite continued fractions:

\[ [3, 7, 15, 1, 292] = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}} \ldots} \]
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Infinite continued fractions:

\[
[a_0, a_1, a_2, a_3, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}
\]
### Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

\[ \sqrt{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots] \]

What about numbers of degree \( \geq 3 \)?

\[ \sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots] \]

Lang and Trotter: "no significant departure from random behaviour"
Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

\[ \sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots] \]

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“Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?”

R. K. Guy, 2004
A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \ldots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$. 

Theorem (arXiv:1506.00695, 2015) If the ZERO PROBLEM is decidable at dimension 9 then 
\[ \{ x \in \mathbb{R} \cap A : S(x) < \infty \} \] is recursively enumerable. 

Remark Perhaps this set is recursive—it may even be $\emptyset$ or $\mathbb{R} \cap A$. However proving recursive enumerability would be a significant achievement.
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Diophantine Approximation

How well can one approximate a real number $x$ with rationals?

$$|x - \frac{m}{n}|$$

Theorem (Dirichlet 1842)

There are infinitely many integers $m, n$ such that

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  has no solutions.

- Relate this to the existence of zeros of order-9 exponential polynomial $f(t)$ with terms $e^{ixt}$ and $e^{it}$. 
## The ZERO Problem

### ZERO Problem

**Instance:** $f$

**Question:** Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

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- Diophantine approximation
  - Kronecker’s Theorem on simultaneous Diophantine approximation.
The ZERO Problem

**Instance:** \( f \)

**Question:** Is there \( t \in \mathbb{R}_{\geq 0} \) such that \( f(t) = 0 \)?

**Theorem (arXiv:1507.03632, 2015)**

*In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.*

- Diophantine approximation
  - Kronecker’s Theorem on simultaneous Diophantine approximation.
  - Baker’s Theorem on lower bounds for linear forms in logarithms of algebraic numbers.
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- Model theory of the reals
  - *o-minimality* of $(\mathbb{R}, <, +, \times, e^x, 0, 1)$. 
Conclusion and Perspectives
The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

for all $n \geq 0$. 

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear: 

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where $F$ is finite and each $A_i$ is a full arithmetic progression.

Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.
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Wrapping Things Up

Continuous Skolem Problem

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Does \( \exists t \text{ such that } f(t) = 0 \)?

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- Even the bounded problem is hard (apparently).
- Formidable “mathematical obstacle” at dimension 9 in the unbounded case.
- The infinite-zeros problem is also hard.
- Diophantine-approximation techniques unavoidable.